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On $H^*(\mathcal{C}; k^\times)$ for fusion systems

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Abstract

We give a cohomological criterion for the existence and uniqueness of solutions of the 2-cocycle gluing problem in block theory. The existence of a solution for the 2-cocycle gluing problem is further reduced to a property of fusion systems of certain finite groups associated with the fusion system of a block.

1 Introduction

Given a block b of a finite group G over an algebraically closed field k of characteristic p with defect group P and associated fusion system \mathcal{F} , there is, for any \mathcal{F} -centric subgroup Q of P , a canonically determined element $\alpha_Q \in H^2(\text{Aut}_{\mathcal{F}}(Q); k^\times)$, by [13, 1.12]. It is conjectured in [14, 4.2] that this family can be glued together to a class $\alpha \in H^2(\mathcal{F}^c; k^\times)$ satisfying $\alpha|_{\text{Aut}_{\mathcal{F}}(Q)} = \alpha_Q$ for any Q belonging to the full subcategory \mathcal{F}^c of \mathcal{F} -centric subgroups of P . We describe a cohomological criterion for the existence and uniqueness of α . This can be formulated more generally for EI -categories. Following [18, 9.2], an EI -category is a small category \mathcal{C} with the property $\text{End}_{\mathcal{C}}(X) = \text{Aut}_{\mathcal{C}}(X)$ for any object X in \mathcal{C} . The set $[\mathcal{C}]$ of isomorphism classes of objects in \mathcal{C} becomes then a partially ordered set via $[X] \leq [Y]$ if $\text{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, for X, Y objects in \mathcal{C} and $[X], [Y]$ their respective isomorphism classes. A morphism $\varphi : X \rightarrow Y$ in \mathcal{C} need not induce a map between the automorphism groups of X, Y ; the *subdivision* of \mathcal{C} is a tool to address this issue; this is the category defined as follows. The objects of $S(\mathcal{C})$ are faithful functors $\sigma : [m] \rightarrow \mathcal{C}$, where m is a nonnegative integer and the totally ordered set $[m] = \{0 < 1 < \dots < m\}$ is viewed as category in the obvious way; a morphism in $S(\mathcal{C})$ from σ to another object $\tau : [n] \rightarrow \mathcal{C}$ is a pair (α, μ) consisting of an injective order preserving map $\alpha : [m] \rightarrow [n]$ and an isomorphism of functors $\mu : \sigma \cong \tau \circ \alpha$. The composition of (α, μ) with another morphism (β, ν) from τ to $\rho : [r] \rightarrow \mathcal{C}$ is defined by $(\beta, \nu) \circ (\alpha, \mu) = (\beta \circ \alpha, (\nu \alpha) \circ \mu)$, where $\nu \alpha : \tau \circ \alpha \cong \rho \circ \beta \circ \alpha$ is induced by precomposing ν with α . Loosely speaking, $S(\mathcal{C})$ consists of chains of non-isomorphisms in \mathcal{C} . It is easy to see that (α, μ) induces a group homomorphism $\text{Aut}_{S(\mathcal{C})}(\tau) \rightarrow \text{Aut}_{S(\mathcal{C})}(\sigma)$ mapping $(\text{Id}_{[n]}, \gamma)$ to $(\text{Id}_{[m]}, \mu^{-1} \circ (\gamma \alpha) \circ \mu)$, for any automorphism γ of the functor τ , where $\gamma \alpha$ is the induced automorphism of $\tau \circ \alpha$. Clearly $S(\mathcal{C})$ is again an EI -category. The partially ordered set $[S(\mathcal{C})]$, viewed as topological space, is called the *orbit space* of \mathcal{C} . We denote by \mathbf{Ab} the category of abelian groups.

Theorem 1.1. *Let \mathcal{C} be a finite EI -category and k an algebraically closed field. For any positive integer i there is a canonical functor $\mathcal{A}^i : [S(\mathcal{C})] \rightarrow \mathbf{Ab}$ sending $[\sigma] \in [S(\mathcal{C})]$ to $H^i(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$. If $H^1([S(\mathcal{C})]; k^\times) = H^2([S(\mathcal{C})]; k^\times) = 0$ then $H^1(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^1)$ and if also $H^3([S(\mathcal{C})]; k^\times) = H^4([S(\mathcal{C})]; k^\times) = 0$ then there is an exact sequence of abelian groups*

$$0 \rightarrow H^1([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^2(\mathcal{C}; k^\times) \rightarrow \lim_{[S(\mathcal{C})]}(\mathcal{A}^2) \rightarrow H^2([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^3(\mathcal{C}; k^\times)$$

In particular, the group $H^2(\mathcal{C}; k^\times)$ is finite, of order coprime to $\text{char}(k)$ if $\text{char}(k)$ is positive.

As mentioned above, the motivation for considering the map $H^2(\mathcal{C}; k^\times) \rightarrow \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$ comes from block theory, which is the reason for stating the above theorem for k^\times , but it is worth noting that the exact sequence in the theorem holds for any abelian group instead of k^\times . In order to be more precise, let b be a block of a finite group G over an algebraically closed field k of positive characteristic p , let P be a defect group of b and \mathcal{F} an associated fusion system on P . Set $\mathcal{C} = \mathcal{F}^c$, the full subcategory \mathcal{F} consisting of all \mathcal{F} -centric subgroups of P . Then \mathcal{C} is a right ideal in \mathcal{F} , and hence $[S(\mathcal{C})]$ is contractible by [16, 1.1]; in particular, $H^i([S(\mathcal{C})]; k^\times) = 0$ for $i > 0$. Thus \mathcal{C} satisfies the hypotheses of Theorem 1.1. As a consequence of work of Külshammer and Puig [13, 1.8, 1.12] in conjunction with Dade's splitting theorem on fusion, the block b determines for every $\sigma \in S(\mathcal{C})$ a class $\alpha_\sigma \in H^2(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$, and the family (α_σ) of these classes determines an element β in $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$. Denote by γ the image of β in $H^2([S(\mathcal{C})]; \mathcal{A}^1)$. Then, by Theorem 1.1 above, the gluing problem [14, 4.2] has a solution if $\gamma = 0$, and the solution is unique if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = 0$. In particular, if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = H^2([S(\mathcal{C})]; \mathcal{A}^1) = 0$ then we have an isomorphism $H^2(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$, and so the gluing problem [14, 4.2] would have a solution for any block b with fusion system \mathcal{F} . This isomorphism holds trivially if $\mathcal{F} = N_{\mathcal{F}}(P)$ (which includes the case where P is abelian) and if P is a tame 2-group, in which case the right side is well-known to be zero, and the left side is zero by a result of S. Park [21]. In general, this isomorphism is relevant for the block theoretic reformulation of Alperin's weight conjecture in terms of Bredon cohomology in [15, 4.3]. The purpose of the next result is to reduce the vanishing of $H^2([S(\mathcal{F}^c)]; \mathcal{A}^1)$ further to a statement on finite groups. If all else fails, this points at least hypothetically to a strategy to solve the existence part of the gluing problem using the classification of finite simple groups. For any finite group G , denote by $\Delta_p(G)$ the partially ordered G -set of chains $\sigma = Q_0 < Q_1 < \dots < Q_m$ of nontrivial p -subgroups Q_i of G and denote by $[\Delta_p(G)]$ the set of G -conjugacy classes of chains in $\Delta_p(G)$, viewed as partially ordered set via taking subchains. Denote by $\mathcal{N}_G : [\Delta_p(G)] \rightarrow \mathbf{Ab}$ the covariant functor sending the G -conjugacy class $[\sigma]$ of the chain $\sigma \in \Delta_p(G)$ as above to the abelian group $\mathcal{N}_G([\sigma]) = \text{Hom}(N_G(\sigma); k^\times)$, where $N_G(\sigma)$ is the intersection of the normalisers $N_G(Q_i)$, $0 \leq i \leq m$. As before, one checks that this is a well-defined functor which does not depend, up to unique isomorphism of functors, on the choice of a representative σ of $[\sigma]$.

Theorem 1.2. *Let \mathcal{F} be a fusion system on a finite p -group P . Suppose that for any finite group G isomorphic to $\text{Aut}_{\mathcal{F}}(Q)/\text{Inn}(Q)$ for some \mathcal{F} -centric subgroup Q of P we have $H^1([\Delta_p(G)]; \mathcal{N}_G) = 0$. Then $H^2([S(\mathcal{F}^c)]; \mathcal{A}^1) = 0$; in particular, the canonical map $H^2(\mathcal{F}^c; k^\times) \rightarrow \lim_{[S(\mathcal{F}^c)]}(\mathcal{A}^2)$ is surjective.*

We do not know whether $H^1([\Delta_p(G)]; \mathcal{N}_G)$ vanishes for all finite groups G and do not have, at this point, enough backup from examples to make this a conjecture. Theorem 1.2 will follow from a spectral sequence using a filtration indexed by isomorphism classes of \mathcal{F} -centric subgroups, by making use of the fact that the cohomology of \mathcal{A}^1 may be calculated using *normal* chains of subgroups of P . After reviewing briefly some basic facts on functor cohomology in section 2 we consider regular EI -categories and prove Theorem 1.1 in section 3. This is followed by a section proving Theorem 1.2 and a section 5 on regular functors between EI -categories, which in turn is used in the last section to show that in order to calculate $H^*(\mathcal{C}; k^\times)$ for a right ideal \mathcal{C} of a fusion system, we may replace \mathcal{C} by its image in the orbit category or its inverse image in a centric linking system.

2 Background

We collect, mostly from [5], [6], [9], [10], [11], [23], [24], background material on functor cohomology, which we will use without further comment. See also [25] for a broader exposition and further references. A *right ideal* in a category \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} with the property that if $\varphi : X \rightarrow Y$ is a morphism in \mathcal{C} and if X belongs to \mathcal{D} , then also Y belongs to \mathcal{D} . For \mathcal{C} a small category denote by $\hat{\mathcal{C}}$

and $\tilde{\mathcal{C}}$ the categories of covariant and contravariant functors from \mathcal{C} to \mathbf{Ab} , respectively. The cohomology $H^*(\mathcal{C}; \mathcal{A})$ or $H^*(\mathcal{C}^{op}; \mathcal{A})$ of a functor \mathcal{A} in $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}$, respectively, is the graded abelian group defined as follows: for any nonnegative integer n denote by $H^n(\mathcal{C}; \mathcal{A})$ or $H^n(\mathcal{C}^{op}; \mathcal{A})$ the n -th right derived functor of the limit functor $\lim_{\mathcal{C}}$ over \mathcal{C} from $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}$ to \mathbf{Ab} , respectively. If A is an abelian group and \mathcal{A} the constant covariant functor on \mathcal{C} taking the value A , we write $H^n(\mathcal{C}; A)$ instead of $H^n(\mathcal{C}; \mathcal{A})$; similarly for the contravariant constant functor. It is well-known that $H^*(\mathcal{C}; A) \cong H^*(\mathcal{C}^{op}; A)$. If $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor, we denote by $\Phi^* : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{C}}$ the induced restriction functor sending \mathcal{A} in $\tilde{\mathcal{C}}$ to $\mathcal{A} \circ \Phi$; we use (abusively) the same notation for the restriction functor $\tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{C}}$. We denote by $\Phi_*, \Phi_! : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ the left and right adjoint (Kan extension functors) of Φ^* , and as before, use the same notation for the categories $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}$. By [24, 1.4.(ii)] or [11, 5.3] or [5, Appendix II, Thm. 3.6] there is a cohomology spectral sequence, called *base change spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}^{op}; R^q\Phi_!(\mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}^{op}; \mathcal{A})$$

for any \mathcal{A} in $\tilde{\mathcal{C}}$, where $R^q\Phi_!$ is the q -th right derived functor of $\Phi_!$. It is well-known (see e.g. [5, Appendix II, §3] for the homology version) that $R^q\Phi_!$ can be computed explicitly by

$$R^q\Phi_!(\mathcal{A})(Y) = H^q((\Phi^Y)^{op}; \mathcal{A}^Y)$$

where \mathcal{A} is in $\tilde{\mathcal{C}}$, Y is an object in \mathcal{D} , where Φ^Y is the category with objects pairs (X, φ) with X an object in \mathcal{C} and $\varphi : \Phi(X) \rightarrow Y$ a morphism in \mathcal{D} ; a morphism in Φ^Y from (X, φ) to (X', φ') is a morphism $\alpha : X \rightarrow X'$ in \mathcal{C} satisfying $\varphi' \circ \Phi(\alpha) = \varphi$, and \mathcal{A}^Y is the functor obtained by restricting \mathcal{A} to Φ^Y via the forgetful functor $\Phi^Y \rightarrow \mathcal{C}$ sending (X, φ) to X . We will need the base change spectral sequence only the case where \mathcal{C} is an *EI*-category and \mathcal{D} a partially ordered set. In that case, for any Y in \mathcal{D} , the category Φ^Y can be identified with the category $\mathcal{C}_{\leq Y}$ consisting of all objects X in \mathcal{C} such that $\text{Hom}_{\mathcal{D}}(\Phi(X), Y)$ is non-empty, and hence the base change spectral sequence takes the well-known form

$$E_2^{p,q} = H^p(\mathcal{D}^{op}; Y \mapsto H^q(\mathcal{C}_{\leq Y}; \mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}^{op}; \mathcal{A})$$

for any \mathcal{A} in $\tilde{\mathcal{C}}$, where in the expression $H^q(\mathcal{C}_{\leq Y}; \mathcal{A})$ the restriction of \mathcal{A} to $\mathcal{C}_{\leq Y}$ is abusively again denoted by \mathcal{A} . If $\Phi_!$ is exact the base change spectral sequence collapses. This happens in particular if Φ has a right adjoint $\Psi : \mathcal{D} \rightarrow \mathcal{C}$, for in that case the right adjoint $\Phi_!$ of Φ^* is isomorphic to the exact restriction functor Ψ^* . Thus, if Φ has a right adjoint we have $H^*(\mathcal{D}^{op}; \Phi_!(\mathcal{A})) \cong H^*(\mathcal{C}^{op}; \mathcal{A})$ for any \mathcal{A} in $\tilde{\mathcal{C}}$ (cf. [9, 3.1]).

3 Regular *EI*-categories and proof of Theorem 1.1

Following [15, 2.1], an *EI*-category \mathcal{C} is called *regular* if for any two objects X, Y in \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ the group $\text{Aut}_{\mathcal{C}}(X)$ acts regularly (i.e. transitively and freely) on $\text{Hom}_{\mathcal{C}}(X, Y)$. Any morphism in a regular *EI*-category is a monomorphism. For any *EI*-category \mathcal{C} the subdivision $S(\mathcal{C})$ is regular, and if \mathcal{C} is regular, there is a contravariant functor from \mathcal{C} to the category of groups sending an object X to its automorphism group $\text{Aut}_{\mathcal{C}}(X)$ and a morphism $\varphi : X \rightarrow Y$ in \mathcal{C} to the unique map $\text{Aut}_{\mathcal{C}}(Y) \rightarrow \text{Aut}_{\mathcal{C}}(X)$ which sends $\sigma \in \text{Aut}_{\mathcal{C}}(Y)$ to the unique $\rho \in \text{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$; we use the regularity of \mathcal{C} for the existence and uniqueness of ρ (cf [15, 2.2]). For any *EI*-category \mathcal{C} , the subdivision $S(\mathcal{C})$ comes with canonical functors from $S(\mathcal{C})$ to \mathcal{C} and \mathcal{C}^{op} sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to $\sigma(m)$ and $\sigma(0)$, respectively. If every isomorphism class of \mathcal{C} has a unique element then $S(\mathcal{C})$ is equivalent to the opposite of the category $s(\mathcal{C})$ defined in [23, §1], and hence [23, 1.5] implies that the canonical functor $S(\mathcal{C}) \rightarrow \mathcal{C}$ induces, for any abelian group A , an isomorphism $H^*(S(\mathcal{C}); A) \cong H^*(\mathcal{C}; A)$. For regular *EI*-categories the appropriate base change spectral sequence specialises to the following spectral sequence:

Theorem 3.1. *Let \mathcal{C} be a regular EI-category and A an abelian group. There is a cohomology spectral sequence $E_2^{p,q} = H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(\text{Aut}_{\mathcal{C}}(X); A)) \Rightarrow H^{p+q}(\mathcal{C}; A)$.*

Proof. Denote by $\Phi : S(\mathcal{C}) \rightarrow [\mathcal{C}]^{op}$ the canonical functor sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to the isomorphism class $[\sigma(0)]$ in $[\mathcal{C}]$. One checks that for any X in \mathcal{C}^{op} we have $S(\mathcal{C})_{\leq [X]} = S(\mathcal{C}_{\geq X})$. Thus the base change spectral sequence associated with Φ takes the form

$$E_2^{p,q} = H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(S(\mathcal{C}_{\geq X}); \mathcal{A})) \Rightarrow H^{p+q}(S(\mathcal{C}); \mathcal{A})$$

As mentioned above, we have $H^{p+q}(S(\mathcal{C}); \mathcal{A}) \cong H^{p+q}(\mathcal{C}; \mathcal{A})$ and $H^q(S(\mathcal{C}_{\geq X}); \mathcal{A}) \cong H^q(\mathcal{C}_{\geq X}; \mathcal{A})$. By 3.3 below, this is isomorphic to $H^q(\text{Aut}_{\mathcal{C}}(X); A)$, whence the result. \square

Lemma 3.2. *Let \mathcal{C} be a small category, X an object in \mathcal{C} and denote by \mathcal{D} the full subcategory of \mathcal{C} having X as unique object. Suppose that for any object Y in \mathcal{C} there is a morphism $\iota_Y : X \rightarrow Y$ in \mathcal{C} such that the map $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ sending $\alpha \in \text{Aut}_{\mathcal{C}}(X)$ to $\iota_Y \circ \alpha$ is a bijection. Then the inclusion functor $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ has a right adjoint $\Psi : \mathcal{C} \rightarrow \mathcal{D}$.*

Proof. Define Ψ on objects by $\Psi(Y) = X$ for all Y in \mathcal{C} . For a morphism $\varphi : Y \rightarrow Z$ in \mathcal{C} , define the morphism $\Psi(\varphi) \in \text{Aut}_{\mathcal{C}}(X) = \text{Aut}_{\mathcal{D}}(X)$ as follows. By the assumptions there is a unique $\beta \in \text{Aut}_{\mathcal{C}}(X)$ such that $\varphi \circ \iota_Y = \iota_Z \circ \beta$. We set $\Psi(\varphi) = \beta$. A trivial verification shows that this construction is functorial and yields a right adjoint for Φ . \square

Lemma 3.3. *Let \mathcal{C} be a regular EI-category, X an object in \mathcal{C} and A an abelian group. Restriction induces an isomorphism $H^*((\mathcal{C}_{\geq X})^{op}; A) \cong H^*(\text{Aut}_{\mathcal{C}}(X); A)$.*

Proof. For any object Y in $\mathcal{C}_{\geq X}$ choose a morphism $\iota_Y : X \rightarrow Y$. Since \mathcal{C} is regular, composition with ι_Y induces a bijection $\text{Aut}_{\mathcal{C}}(X) \cong \text{Hom}_{\mathcal{C}}(X, Y)$. Thus 3.2 applies, showing that the inclusion functor $\{X\} \rightarrow \mathcal{C}_{\geq X}$ has a right adjoint Ψ . But then the base change spectral sequence associated with this inclusion functor collapses and yields the isomorphism as stated. \square

Proof of Theorem 1.1. We use the notation of 1.1. The spectral sequence 3.1 applied to the regular EI-category $S(\mathcal{C})$ and the abelian group k^\times takes the form

$$E_2^{p,q} = H^p([S(\mathcal{C})]; \mathcal{A}^q) \Rightarrow H^{p+q}(S(\mathcal{C}); k^\times)$$

That is, the E_2 -page has the form

$$\begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ H^0([S(\mathcal{C})]; \mathcal{A}^2) & H^1([S(\mathcal{C})]; \mathcal{A}^2) & H^2([S(\mathcal{C})]; \mathcal{A}^2) & H^3([S(\mathcal{C})]; \mathcal{A}^2) & \cdots & \cdots \\ H^0([S(\mathcal{C})]; \mathcal{A}^1) & H^1([S(\mathcal{C})]; \mathcal{A}^1) & H^2([S(\mathcal{C})]; \mathcal{A}^1) & H^3([S(\mathcal{C})]; \mathcal{A}^1) & \cdots & \cdots \\ k^\times & H^1([S(\mathcal{C})]; k^\times) & H^2([S(\mathcal{C})]; k^\times) & H^3([S(\mathcal{C})]; k^\times) & \cdots & \cdots \end{array}$$

This spectral sequence approximates $H^*(S(\mathcal{C}); k^\times) \cong H^*(\mathcal{C}; k^\times)$. Thus if the groups $H^1([S(\mathcal{C})]; k^\times)$ and $H^2([S(\mathcal{C})]; k^\times)$ are zero there is no differential starting or ending at the coordinates $(0, 1)$, and hence we get the isomorphism $H^1(\mathcal{C}; k^\times) \cong H^0([S(\mathcal{C})]; \mathcal{A}^1) = \lim_{[S(\mathcal{C})]} (\mathcal{A}^1)$ as stated in 1.1. Suppose now in addition that also $H^3([S(\mathcal{C})]; k^\times) = H^4([S(\mathcal{C})]; k^\times) = 0$. There is no nonzero differential starting or ending at $E_2^{1,1} = H^1([S(\mathcal{C})]; \mathcal{A}^1)$, and hence, again since this spectral sequence approximates $H^*(\mathcal{C}; k^\times)$, we get an injective map $H^1([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^2(\mathcal{C}; k^\times)$. The cokernel of this map is the kernel of the differential

$$E_2^{0,2} = \lim_{[S(\mathcal{C})]} (\mathcal{A}^2) \longrightarrow E_2^{2,1} = H^2([S(\mathcal{C})]; \mathcal{A}^1)$$

because all differentials starting at $(0, 2)$ from page 3 onwards are zero by the assumptions. Since from page 3 onwards there is no nonzero differential ending or starting at $(2, 1)$ it follows that the cokernel of the last map is a subgroup of $H^3(\mathcal{C}; k^\times)$. This proves the exactness of the sequence stated in 1.1, and as pointed out earlier, this part of the argument is valid for any abelian group instead of k^\times . The groups $\mathcal{A}^1([\sigma]) \cong \text{Hom}(\text{Aut}_{S(\mathcal{C})}(\sigma), k^\times)$ and $\mathcal{A}^2([\sigma]) = H^2(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$ are finite, of order coprime to $\text{char}(k)$ if $\text{char}(k)$ is positive, hence $H^1([S(\mathcal{C})]; \mathcal{A}^1)$ and $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$ are finite, proving the finiteness of $H^2(\mathcal{C}; k^\times)$ with the properties as stated. \square

4 Proof of Theorem 1.2

The terminology on fusion systems we use follows [17]; in particular, a fusion system means a saturated fusion system in the sense of [3]. See [22, §2] or [3, Appendix] or [17, §3] for details regarding normalisers, centralisers and quotients in fusion systems. For \mathcal{C} a right ideal in a fusion system \mathcal{F} on a finite p -group P denote by $S_{<}(\mathcal{C})$ the full subcategory of all $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ such that for $0 \leq i < j \leq m$ we have $\sigma(i) < \sigma(j)$ and the morphism $\sigma(i < j)$ from $\sigma(i)$ to $\sigma(j)$ is the inclusion map. By [16, 4.2], the subcategory $S_{<}(\mathcal{C})$ is equivalent to $S(\mathcal{C})$. We denote by $S_{\triangleleft}(\mathcal{C})$ the full subcategory of all $\sigma : [m] \rightarrow \mathcal{C}$ in $S_{<}(\mathcal{C})$ such that for $0 \leq i \leq j \leq m$ the subgroup $\sigma(j)$ is normal in $\sigma(i)$, or equivalently, such that $\sigma(i)$ is normal in the maximal subgroup $\sigma(m)$ of the chain of subgroups σ . The proof of 1.2 is based on the following spectral sequence.

Theorem 4.1. *Let \mathcal{F} be a fusion system on a finite p -group P and \mathcal{C} a right ideal in \mathcal{F} . Let $\{R_q \mid 0 \leq q \leq r\}$ be a set of representatives of the \mathcal{F} -isomorphism classes of subgroups of P belonging to \mathcal{C} such that R_q is fully \mathcal{F} -normalised for $0 \leq q \leq r$ and such that $|R_q| \geq |R_{q+1}|$ for $1 \leq q \leq r$. Denote by \mathcal{C}_q the right ideal in $N_{\mathcal{F}}(R_q)/R_q$ consisting of all non trivial subgroups of $N_P(R_q)/R_q$, where $1 \leq q \leq r$. Let $\mathcal{A} : [S_{\triangleleft}(\mathcal{C})] \rightarrow \mathbf{Ab}$ be a covariant functor. Then, for $1 \leq q \leq r$, the functor \mathcal{A} induces a covariant functor $\mathcal{N}^q : [S_{\triangleleft}(\mathcal{C}_q)] \rightarrow \mathbf{Ab}$ and there is a spectral sequence*

$$E_1^{p,q} \Rightarrow H^{p+q}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A})$$

with the following properties.

- (i) We have $E_1^{p,q} = H^{p+q-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$ for $p+q \geq 2$ and $1 \leq q \leq r$.
- (ii) We have $E_1^{p,q} = \{0\}$ if $q < 0$ or $q > r$ or $p+q < 0$, we have $E_1^{p,0} = \{0\}$ for $p \neq 0$, and we have $E_1^{0,0} = \mathcal{A}([P])$.
- (iii) If there is an integer $m \geq 1$ such that $H^m([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q) = \{0\}$ for $1 \leq q \leq r$ then $H^{m+1}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A}) = \{0\}$.
- (iv) If \mathcal{A} vanishes on all chains of length zero in $[S_{\triangleleft}(\mathcal{C})]$ then also $E_1^{p,q} = H^0([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$ for $p+q = 1$ and $1 \leq q \leq r$, and $E_1^{p,q} = \{0\}$ if $p+q = 0$ or $q = 0$.

We break up the proof in several steps. Let \mathcal{F} be a fusion system on a finite p -group P , where p is a prime, and let \mathcal{C} be a right ideal in \mathcal{F} . Following the notation in [16, 4.1] the category $S_{\triangleleft}(\mathcal{C})$ is the full subcategory of $S(\mathcal{C})$ whose elements can be denoted as chains

$$\sigma = Q_0 < Q_1 < \cdots < Q_m$$

of subgroups Q_i of P belonging to \mathcal{C} such that Q_i is normal in Q_m for $0 \leq i \leq m$. Chains of this type were introduced in [12]. Given such a chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$ we denote, for any integer i such that $0 \leq i \leq m$, by $\sigma \setminus i$ the chain

$$\sigma \setminus i = Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m$$

obtained from removing Q_i . Inclusion of chains yields a canonical morphism $\sigma \setminus i \longrightarrow \sigma$ in $S_{\triangleleft}(\mathcal{C})$. For any fully \mathcal{F} -normalised subgroup R of P in \mathcal{C} we denote by $S_{R\triangleleft}(\mathcal{C})$ the full subcategory of $S_{\triangleleft}(\mathcal{C})$ consisting of all chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ with $Q_0 \cong R$ in \mathcal{F} and $m \geq 1$. We denote by \mathcal{C}_R the right ideal in $N_{\mathcal{F}}(R)/R$ consisting of all non-trivial subgroups of $N_P(R)/R$. We keep this notation throughout this section.

Lemma 4.2. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain in $S_{\triangleleft}(\mathcal{F})$ and let R be a fully \mathcal{F} -normalised subgroup of P such that $R \cong Q_0$ in \mathcal{F} . Then there is a chain $\tau = (R_0 < R_1 < \cdots < R_m)$ in $S_{\triangleleft}(\mathcal{F})$ such that $\tau \cong \sigma$ and $R_0 = R$.*

Proof. By standard properties of fusion systems [17, 2.6] there is a morphism $\varphi : N_P(Q_0) \rightarrow P$ such that $\varphi(Q_0) = R$, because R is fully \mathcal{F} -normalised. Taking $\tau = \varphi(\sigma) = (\varphi(Q_0) < \varphi(Q_1) < \cdots < \varphi(Q_m))$ proves the lemma. \square

Lemma 4.3. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ and $\tau = (R_0 < R_1 < \cdots < R_m)$ be chains in $S_{\triangleleft}(\mathcal{F})$ such that $Q = Q_0 = R_0$ is fully \mathcal{F} -normalised. Suppose that $m > 0$. Then $\sigma \cong \tau$ in $S_{\triangleleft}(\mathcal{F})$ if and only if $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$.*

Proof. Any isomorphism $\sigma \cong \tau$ induces an automorphism on $Q = Q_0 = R_0$, and hence an isomorphism $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$. Conversely, any isomorphism $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$ induces an automorphism on Q , hence an isomorphism $\sigma \cong \tau$. \square

Lemma 4.4. *Suppose that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ and $\tau = (R_0 < R_1 < \cdots < R_m)$ be two chains in $S_{\triangleleft}(\mathcal{F})$ such that $Q \subseteq Q_0$ and $Q \subseteq R_0$. Then the chains $\bar{\sigma} = (Q_0/Q < Q_1/Q < \cdots < Q_m/Q)$ and $\bar{\tau} = (R_0/Q < R_1/Q < \cdots < R_m/Q)$ belong to the category $S_{\triangleleft}(\mathcal{F}/Q)$, and we have $\sigma \cong \tau$ in $S_{\triangleleft}(\mathcal{F})$ if and only if $\bar{\sigma} \cong \bar{\tau}$ in $S_{\triangleleft}(\mathcal{F}/Q)$.*

Proof. Clearly the chains $\bar{\sigma}, \bar{\tau}$ are in $S_{\triangleleft}(\mathcal{F}/Q)$, and if $\sigma \cong \tau$ then $\bar{\sigma} \cong \bar{\tau}$. Suppose conversely that we have an isomorphism $\bar{\mu} = (\bar{\mu}_i)_{0 \leq i \leq m} : \bar{\sigma} \cong \bar{\tau}$ given by a family of isomorphisms $\bar{\mu}_i : Q_i/Q \cong R_i/Q$ in \mathcal{F}/Q . Then any representative μ_m of $\bar{\mu}_m$ is an isomorphism $Q_m \cong R_m$ which sends Q_i to R_i , for $0 \leq i \leq m$, and hence setting $\mu_i = \mu_m|_{Q_i}$ yields an isomorphism $\sigma \cong \tau$. \square

Proposition 4.5. *Let R be a fully \mathcal{F} -normalised subgroup of P . The map sending a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of positive length m in $S_{R\triangleleft}(\mathcal{C})$ to the chain $\bar{\sigma} \setminus 0 = (Q_1/R < Q_2/R < \cdots < Q_m/R)$ in $S_{\triangleleft}(\mathcal{C}_R)$ induces an isomorphism of posets $[S_{R\triangleleft}(\mathcal{C})] \cong [S_{\triangleleft}(\mathcal{C}_R)]$.*

Proof. This follows from combining the three previous Lemmas. \square

Proof of Theorem 4.1. We use the notation from the statement of 4.1; in particular, $\{R_q \mid 0 \leq q \leq r\}$ is a system of representatives of the isomorphism classes in \mathcal{C} with all R_q fully \mathcal{F} -normalised and ordered in such a way that $|R_q| \geq |R_{q+1}|$ for $0 \leq q < r$. Note that since \mathcal{C} is a right ideal in \mathcal{F} this implies in particular that $R_0 = P$. By [16, 3.7] applied to the left ideal $[S_{\triangleleft}(\mathcal{C})]$ in $[S(\mathcal{C})]$, the cohomology of \mathcal{A} is that of a cochain complex of abelian groups $C_{\triangleleft} = C_{\triangleleft}(\mathcal{A})$ which is in degree $n \geq 0$ equal to

$$C_{\triangleleft}^n = \bigoplus_{[\sigma] \in [S_{\triangleleft}(\mathcal{C})], |\sigma|=n} \mathcal{A}([\sigma]) .$$

The differential of this complex is an alternating sum of maps $\mathcal{A}([\sigma] < [\tau]) : \mathcal{A}([\sigma]) \rightarrow \mathcal{A}([\tau])$, where $[\sigma], [\tau]$ are isomorphism classes of chains σ, τ in $S_{\triangleleft}(\mathcal{C})$ such that $[\sigma] < [\tau]$ and such that $|\sigma| + 1 = |\tau|$. For any integer q such that $0 \leq q \leq r$ we define a full subcategory $S_{\triangleleft}^{(q)}(\mathcal{C})$ of $S_{\triangleleft}(\mathcal{C})$ as follows: a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_{\triangleleft}(\mathcal{C})$ belongs to $S_{\triangleleft}^{(q)}(\mathcal{C})$ if and only if $Q_0 \cong R_j$ for some $j \geq q$. The subcategory $S_{\triangleleft}^{(q)}(\mathcal{C})$ is in fact a right ideal in $S_{\triangleleft}(\mathcal{C})$: if there is a morphism from a chain

$\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_{\triangleleft}^{(q)}(\mathcal{C})$ to a chain $\sigma' = (Q'_0 < Q'_1 < \cdots < Q'_{m'})$ in $S_{\triangleleft}(\mathcal{C})$ then in particular either $Q_0 \cong Q'_0$ or $|Q'_0| < |Q_0|$. Since $Q_0 \cong R_j$ for some $j \geq q$ this implies $Q'_0 \cong R_{j'}$ for some $j' \geq j$. Thus σ' belongs to the category $S_{\triangleleft}^{(q)}(\mathcal{C})$ as well. It follows that for $0 \leq q \leq r$ there is a subcomplex $C_{\triangleleft}^{(q)}$ of C_{\triangleleft} defined by

$$(C_{\triangleleft}^{(q)})^n = \bigoplus_{\sigma \in [S_{\triangleleft}^{(q)}(\mathcal{C})], |\sigma|=n} \mathcal{A}([\sigma])$$

for $n \geq 0$. Since the category $S_{\triangleleft}(\mathcal{C})$ is filtered by the subcategories

$$S_{\triangleleft}^{(r)}(\mathcal{C}) \subseteq S_{\triangleleft}^{(r-1)}(\mathcal{C}) \subseteq \cdots \subseteq S_{\triangleleft}^{(0)}(\mathcal{C}) = S_{\triangleleft}(\mathcal{C})$$

it follows that the cochain complex C_{\triangleleft} has a filtration of the form

$$\{0\} \subseteq C_{\triangleleft}^{(r)} \subseteq C_{\triangleleft}^{(r-1)} \subseteq \cdots \subseteq C_{\triangleleft}^{(0)} = C_{\triangleleft}.$$

For notational convenience we set $C_{\triangleleft}^{(q)} = 0$ for $q > r$. By [19, 2.6], the spectral sequence associated by this filtration takes the form

$$E_1^{p,q} = H^{p+q}(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \Rightarrow H^{p+q}(C_{\triangleleft}).$$

The quotient complexes of this filtration are zero unless $0 \leq q \leq r$, and they vanish in negative degrees; thus we get $E_1^{p,q} = \{0\}$ for $q < 0$ or $q > r$ or $p+q < 0$ as claimed in (ii). The right side in this spectral sequence is $H^{p+q}(C_{\triangleleft}) = H^{p+q}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A})$ by [16, 3.7] applied to the left ideal $[S_{\triangleleft}(\mathcal{C})]$ in $[S(\mathcal{C})]$. We need to identify $E_1^{p,q}$ in the remaining cases. For $1 \leq q \leq r$ we set $\mathcal{D}_q = S_{R_q \triangleleft}(\mathcal{C})$, the full subcategory of $S_{\triangleleft}(\mathcal{C})$ consisting of all chains $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of positive length m such that $Q_0 \cong R_q$. By convention, \mathcal{D}_0 is the empty category. That is, on object sets, we have a disjoint union of full subcategories

$$S_{\triangleleft}^{(q)}(\mathcal{C}) = S_{\triangleleft}^{(q+1)}(\mathcal{C}) \cup \mathcal{D}_q \cup [R_q].$$

Thus, for $0 \leq q \leq r$, the quotient complex

$$C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}$$

is equal to

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong \bigoplus_{[\sigma] \in [\mathcal{D}_q], |\sigma|=n} \mathcal{A}([\sigma])$$

for $n > 0$, and

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^0 \cong \mathcal{A}([R_q])$$

with $[R_q]$ viewed as isomorphism class of chains of length zero. Note that by our conventions, for $q = 0$ this quotient complex is concentrated in degree zero, where it is equal to $\mathcal{A}([P])$. Thus in particular $E_1^{p,0} = \{0\}$ for $p \neq 0$ and $E_1^{0,0} = \mathcal{A}([P])$ which completes the proof of (ii). Set $\mathcal{C}_q = \mathcal{C}_{R_q}$ for $1 \leq q \leq r$; that is, \mathcal{C}_q is the full subcategory of $N_{\mathcal{F}}(R_q)/R_q$ consisting of all non trivial subgroups of $N_P(R_q)/R_q$; by convention, \mathcal{C}_0 is empty. By Proposition 4.5 we have an isomorphism of posets $[\mathcal{D}_q] \cong [S_{\triangleleft}(\mathcal{C}_{R_q})]$, where $1 \leq q \leq r$. This isomorphism induces a functor

$$\mathcal{N}^q : [S_{\triangleleft}(\mathcal{C}_q)] \longrightarrow \mathbf{Ab}$$

such that

$$\mathcal{N}^q([Q_1/R_q < Q_2/R_q < \cdots < Q_m/R_q]) = \mathcal{A}([R_q < Q_1 < Q_2 < \cdots < Q_m]),$$

where $Q_1 < Q_2 < \cdots < Q_m$ is a chain in $S_{\triangleleft}(N_{\mathcal{F}}(R_q))$ with $R_q < Q_1$ and $m \geq 1$. Note that the isomorphism of posets $[\mathcal{D}_q] \cong [S_{\triangleleft}(\mathcal{C}_q)]$ sends a chain of positive length m to a chain of length $m-1$;

this accounts for the degree shift in the statement of 4.1. Thus we can reformulate our description of the quotient complex $C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}$ as follows: in degree $n > 0$ we have

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong \bigoplus_{[\tau] \in [S_{\triangleleft}(\mathcal{C}_q)], |\tau|=n-1} \mathcal{N}^q([\tau]),$$

(which by our conventions is zero if $q = 0$ and $n > 0$) and degree zero term equal to $\mathcal{A}([R_q])$, as pointed out before. It follows again from the description of the cohomology of posets associated with subdivisions in [16, §3] that for $n > 0$ we have

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong C_{\triangleleft}(\mathcal{N}^q)^{n-1},$$

where $C_{\triangleleft}(\mathcal{N}^q)$ is the complex which computes the cohomology of the functor \mathcal{N}^q as in [16, 3.1], with \mathcal{C}_q instead of \mathcal{C} . In other words, $(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})$ is the mapping cone of a chain map of the form $\mathcal{A}([R_q]) \rightarrow C_{\triangleleft}(\mathcal{N}^q)$, with $\mathcal{A}([R_q])$ viewed as complex concentrated in degree zero. In particular, we have

$$H^n(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \cong H^{n-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$$

for $n \geq 2$ and $1 \leq q \leq r$. This completes the proof of (i). Furthermore, if all functors \mathcal{N}^q , for $1 \leq q \leq r$, have vanishing cohomology in a fixed positive degree m , then the cohomology of C_{\triangleleft} vanishes in degree $m + 1$. Statement (iii) follows. Finally, if \mathcal{A} vanishes on chains of length zero, then

$$C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)} = C_{\triangleleft}(\mathcal{N}^q)[-1]$$

where $0 \leq q \leq r$, and so in this case we get that

$$H^n(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \cong H^{n-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$$

for all integers n . This completes the proof of (iv). \square

Proof of Theorem 1.2. Let \mathcal{F} be a fusion system on a finite p -group P , set $\mathcal{C} = \mathcal{F}^c$ and denote by $\{R_q \mid 0 \leq q \leq r\}$ a system of representatives of the isomorphism classes in \mathcal{C} with all R_q fully \mathcal{F} -normalised and ordered in such a way that $|R_q| \geq |R_{q+1}|$ for $0 \leq q < r$. The plan is to apply 4.1 to the functor \mathcal{A}^1 restricted to $S_{\triangleleft}(\mathcal{C})$. Note that by [16, 4.7, 4.11] we have an isomorphism

$$H^*([S(\mathcal{C})]; \mathcal{A}^1) \cong H^*([S_{\triangleleft}(\mathcal{C})]; \mathcal{A}^1)$$

where we use the same notation \mathcal{A}^1 for the restriction of \mathcal{A}^1 to $S_{\triangleleft}(\mathcal{C})$. Note further that $\mathcal{A}^1([\sigma]) = H^1(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times) = \text{Hom}(\text{Aut}_{S(\mathcal{C})}; k^\times)$ for any σ in $S(\mathcal{C})$. Similarly, for any finite group G the cohomology of the functor \mathcal{N}_G on $[\Delta_p(G)]$ is invariant under restriction to the subset of normal chains in $\Delta_p(G)$. Moreover, $[\Delta_p(G)]$ is isomorphic, as partially ordered set, to $[S(\mathcal{F}_S(G))]$, where S is a Sylow- p -subgroup of G . By [1, Prop. C], for $0 \leq q \leq r$ there is a finite group L_q such that L_q has $N_P(R_q)$ as Sylow- p -subgroup, $R_q = O_p(L_q)$, $C_{L_q}(R_q) = Z(R_q)$ and $N_{\mathcal{F}}(R_q) = \mathcal{F}_{R_q}(L_q)$. Thus $N_{\mathcal{F}}(R_q)/R_q$ is the fusion system of the finite group $L_q/R_q \cong \text{Aut}_{\mathcal{F}}(R_q)/\text{Inn}(R_q)$ on $N_P(R_q)/R_q$. It follows that if \mathcal{C}_q is the right ideal in $N_{\mathcal{F}}(R_q)/R_q$ as defined in the statement of 4.1, then $S_{\triangleleft}(\mathcal{C}_q)$ can be identified with the partially ordered subset of normal chains in $\Delta_p(L_q/R_q)$, and the cohomology of the functor \mathcal{N}_{L_q/R_q} remains invariant under restriction to $[S_{\triangleleft}(\mathcal{C}_q)]$, and this restriction coincides with the functor \mathcal{N}^q in the statement of 4.1. Thus statement (iii) in 4.1 implies Theorem 1.2. \square

5 Regular functors

Definition 5.1. Let \mathcal{C}, \mathcal{D} be *EI*-categories. A covariant functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is called *regular* if Φ induces an isomorphism $[\mathcal{C}] \cong [\mathcal{D}]$, for any two objects X, Y in \mathcal{C} the map from $\text{Hom}_{\mathcal{C}}(X, Y)$ to $\text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(Y))$ induced by Φ is surjective, and for any two objects X, Y in \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, the group $K(X) = \ker(\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X)))$ acts freely on $\text{Hom}_{\mathcal{C}}(X, Y)$ through composition of morphisms and induces a bijection

$$\text{Hom}_{\mathcal{C}}(X, Y)/K(X) \cong \text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(Y)) .$$

If $\mathcal{C} \rightarrow \mathcal{D}$ is an extension of \mathcal{D} by a functor $\mathcal{Z} : \mathcal{D}^{op} \rightarrow \mathbf{Ab}$ then the structural functor $\mathcal{C} \rightarrow \mathcal{D}$ is regular. Clearly an *EI*-category is regular if and only if the canonical functor $\mathcal{C} \rightarrow [\mathcal{C}]$ is regular. The following lemma on lifting commutative diagrams through regular functors is used below to show that regular functors induce regular functors on subdivisions.

Lemma 5.2. *Let \mathcal{C}, \mathcal{D} be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. If*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \mu \downarrow & & \downarrow \nu \\ V & \xrightarrow{\psi} & W \end{array}$$

is a diagram in \mathcal{C} such that the diagram in \mathcal{D}

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(\varphi)} & \Phi(Y) \\ \Phi(\mu) \downarrow & & \downarrow \Phi(\nu) \\ \Phi(V) & \xrightarrow{\Phi(\psi)} & \Phi(W) \end{array}$$

is commutative, then there is a unique automorphism $\rho \in \text{Aut}_{\mathcal{C}}(X)$ such that $\Phi(\rho) = \text{Id}_{\Phi(X)}$ and such that the diagram in \mathcal{C}

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \mu \circ \rho \downarrow & & \downarrow \nu \\ V & \xrightarrow{\psi} & W \end{array}$$

is commutative.

Proof. The morphisms $\nu \circ \varphi$ and $\psi \circ \mu$ are two morphisms from X to W in \mathcal{C} whose images in the morphism set $\text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$ are equal. Since the kernel $K(X)$ of the map $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X))$ acts freely on $\text{Hom}_{\mathcal{C}}(X, Y)$, inducing a bijection $\text{Hom}_{\mathcal{C}}(X, W)/K(X) \cong \text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$ there is a unique $\rho \in K(X)$ with the required property. \square

Proposition 5.3. *Let \mathcal{C}, \mathcal{D} be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Then Φ induces a regular functor $S(\mathcal{C}) \rightarrow S(\mathcal{D})$.*

Proof. There is an obvious functor $S(\mathcal{C}) \rightarrow S(\mathcal{D})$ sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to the object $\Phi \circ \sigma : [m] \rightarrow \mathcal{D}$ in $S(\mathcal{D})$. Set $\bar{\sigma} = \Phi \circ \sigma$ and $|\sigma| = m$. Let $\tau : n \rightarrow \mathcal{C}$ be another object in $S(\mathcal{C})$. We

show that the map $\text{Hom}_{S(\mathcal{C})}(\sigma, \tau) \rightarrow \text{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ induced by Φ is surjective; we proceed by induction over $|\sigma|$. For $|\sigma| = 0$ this follows from the regularity of the functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$. Suppose that $|\sigma| = m$ is positive; denote by σ' the object in $S(\mathcal{C})$ obtained by deleting $\sigma(0)$. Then $|\sigma'| = m-1$, hence by induction, the map from $\text{Hom}_{S(\mathcal{C})}(\sigma', \tau)$ to $\text{Hom}_{S(\mathcal{D})}(\bar{\sigma}', \bar{\tau})$ is surjective. Using the previous lemma one sees that the map from $\text{Hom}_{S(\mathcal{C})}(\sigma, \tau)$ to $\text{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ is surjective as well. The rest is an easy verification. \square

Lemma 5.4. *Let \mathcal{C}, \mathcal{D} be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Let X be an object in \mathcal{C} and set $Y = \Phi(X)$.*

- (i) *The partially ordered set $[\Phi^Y]$ has $[(X, \text{Id}_Y)]$ as unique maximal element.*
- (ii) *We have $\text{Aut}_{\Phi^Y}(X, \text{Id}_Y) = \ker(\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(Y))$.*

Proof. For (i), let (Z, ψ) be an object in Φ^Y ; that is, Z is an object in \mathcal{C} and $\psi : \Phi(Z) \rightarrow Y$ is a morphism in \mathcal{D} . Since Φ is regular, there is a morphism $\beta : Z \rightarrow X$ such that $\Phi(\beta) = \psi = \text{Id}_Y \circ \psi$. Thus β is a morphism in Φ^Y from (Z, ψ) to (X, Id_Y) . In other words, $[(Z, \psi)] \leq [(X, \text{Id}_Y)]$ for any object (Z, ψ) in Φ^Y . This proves (i), and (ii) is trivial. \square

Lemma 5.5. *Let \mathcal{C}, \mathcal{D} be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Let X be an object in \mathcal{C} and set $Y = \Phi(X)$. Suppose that every morphism in \mathcal{C} is a monomorphism.*

- (i) *The category Φ^Y is regular.*
- (ii) *For any object (Z, ψ) in Φ^Y we have $\text{Aut}_{\Phi^Y}(Z, \psi) = \ker(\text{Aut}_{\mathcal{C}}(Z) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(Z)))$.*

Proof. Let $\alpha, \beta : (Z, \psi) \rightarrow (Z', \psi')$ be two morphisms in Φ^Y . That is, Z, Z' are objects in \mathcal{C} and $\psi \in \text{Hom}_{\mathcal{D}}(\Phi(Z), Y)$, $\psi' \in \text{Hom}_{\mathcal{D}}(\Phi(Z'), Y)$ such that $\psi' \circ \Phi(\alpha) = \psi = \psi' \circ \Phi(\beta)$. Since Φ is regular there are morphisms $\varphi \in \text{Hom}_{\mathcal{C}}(Z, X)$, $\varphi' \in \text{Hom}_{\mathcal{C}}(Z', X)$ such that $\Phi(\varphi) = \psi$ and $\Phi(\varphi') = \psi'$. Then $\Phi(\varphi' \circ \alpha) = \Phi(\varphi) = \Phi(\varphi' \circ \beta)$. The regularity of Φ implies that there is a unique automorphism $\gamma \in \text{Aut}_{\mathcal{C}}(X)$ such that $\varphi' \circ \alpha \circ \gamma = \varphi' \circ \beta$ and such that $\Phi(\gamma) = \text{Id}_Y$. Since φ' is a monomorphism we get that $\alpha \circ \gamma = \beta$. It follows that γ is the unique automorphism of (Z, ψ) in Φ^Y satisfying $\alpha \circ \gamma = \beta$. This proves (i). When applied to the case $(Z, \psi) = (Z', \psi')$ and $\beta = \text{Id}_Z$ this argument proves (ii). \square

Theorem 5.6. *Let \mathcal{C}, \mathcal{D} be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Suppose that every morphism in \mathcal{C} is a monomorphism. Let A be an abelian group and for any object X in \mathcal{C} denote by $K(X)$ the kernel of the group homomorphism $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X))$ induced by Φ . Suppose that $H^q(K(X); A) = \{0\}$ for $q \geq 0$ and any object X in \mathcal{C} , with respect to the trivial action of $K(X)$ on A . Then Φ induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$.*

Proof. Let Y be an object in \mathcal{D} . By 5.5, the category Φ^Y is regular, and by 5.4, the partially ordered set $[\Phi^Y]$ has a unique maximal element. By 3.1 applied to Φ^Y there is a spectral sequence of the form

$$H^p([\Phi^Y]; [(Z, \psi)]) \mapsto H^q(\text{Aut}_{\Phi^Y}(Z, \psi); A) \Rightarrow H^{p+q}(\Phi^Y; A)$$

Now $\text{Aut}_{\Phi^Y}(Z, \psi) = K(Z)$ by 5.5, hence the assumptions imply that this spectral sequence collapses to an isomorphism $H^p([\Phi^Y]; A) \cong H^p(\Phi^Y; A)$. But this group is zero for q positive as $[\Phi^Y]$ has unique maximal element. This shows that $R^q\Phi_!(A) = 0$ for q positive, and hence the base change spectral sequence of Φ collapses to an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$ as stated. \square

6 Further invariance properties of $H^*(\mathcal{C}; k^\times)$

Let p be a prime. A fusion system \mathcal{F} of a block with defect group P of a finite p -solvable group G is, by a result of Puig, always the fusion system of a finite group L having P as Sylow- p -subgroup such that $Q = O_p(L)$ is an \mathcal{F} -centric subgroup of P . The results of this section, besides providing some reduction techniques for calculating $H^*(\mathcal{F}^c; k^\times)$, imply that $H^2(\mathcal{F}^c; k^\times)$ is the p' -part of the Schur multiplier of L in that case (see 6.6 below). It is well-known that any element α in the p' -part of the Schur multiplier of L arises as Külshammer-Puig 2-cocycle of the fusion system \mathcal{F} at Q in a suitable block of a finite central p' -extension \hat{L} of L determined by α . We first show the invariance of $H^*(\mathcal{F}^c; k^\times)$ with respect to taking quotients by central p -subgroups. If \mathcal{F} is a fusion system on a finite p -group P and Z is a subgroup of $Z(P)$ then the equality $\mathcal{F} = C_{\mathcal{F}}(Z)$ means that every morphism $\varphi : Q \rightarrow R$ in \mathcal{F} can be extended to a morphism $\psi : QZ \rightarrow RZ$ in \mathcal{F} such that $\psi|_Z = \text{Id}_Z$. If $\mathcal{F} = C_{\mathcal{F}}(Z)$ then \mathcal{F} induces a fusion system on P/Z , denoted by \mathcal{F}/Z . Given a fusion system \mathcal{F} on a finite p -group P , we denote as before by \mathcal{F}^c the full subcategory in \mathcal{F} of all \mathcal{F} -centric subgroups of P ; this is a right ideal in \mathcal{F} . We denote by $\bar{\mathcal{F}}$ the quotient category of \mathcal{F} having as objects the subgroups of P and as morphism sets the orbits $\text{Aut}_R(R) \backslash \text{Hom}_{\mathcal{F}}(Q, R)$ of the morphism set $\text{Hom}_{\mathcal{F}}(Q, R)$ with respect to the action of the group of inner automorphisms $\text{Aut}_R(R)$ of R via composition of group homomorphisms, called the *orbit category* of \mathcal{F} . Denote by $\bar{\mathcal{F}}^c$ the image of \mathcal{F}^c in $\bar{\mathcal{F}}$. While all morphisms in \mathcal{F} are monomorphisms, this is no longer true in $\bar{\mathcal{F}}$; however, all morphisms in $\bar{\mathcal{F}}^c$ are epimorphisms. For the centric linking system \mathcal{L} of \mathcal{F} we refer to [3, 1.7]. The following invariance properties are essentially consequences of 5.6.

Proposition 6.1. *Let \mathcal{F} be a fusion system on a finite p -group P such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ for some subgroup Z of $Z(P)$ and let \mathcal{C} be a right ideal in \mathcal{F} consisting of subgroups of P containing Z such that the image \mathcal{C}/Z of \mathcal{C} in \mathcal{F}/Z is contained in $(\mathcal{F}/Z)^c$. Let k be an algebraically closed field of characteristic p . The canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is regular and induces an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\mathcal{C}/Z; k^\times)$.*

Proof. Let Q, R be subgroups of P belonging to \mathcal{C} . Then Z is contained in Q, R by the assumptions. The morphism set $\text{Hom}_{\mathcal{F}/Z}(Q/Z, R/Z)$ is the canonical image of the morphism set $\text{Hom}_{\mathcal{F}}(Q, R)$; in particular, the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is surjective on morphisms. This implies also that if $Q/Z \cong R/Z$ in \mathcal{C}/Z then $Q \cong R$ in \mathcal{C} , and hence we have the isomorphism of posets $[\mathcal{C}] \cong [\mathcal{C}/Z]$. If $\varphi, \psi : Q \rightarrow R$ are two morphisms in \mathcal{C} whose images in \mathcal{C}/Z are equal, then in particular, $\varphi(Q) = \psi(Q) \subseteq R$, and with the obvious abuse of notation we get an automorphism $\kappa = \psi^{-1} \circ \varphi$ of Q whose image in \mathcal{C}/Z is the identity on Q/Z . Thus κ is the unique element of the group $K(Q) = \ker(\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z))$ satisfying $\varphi = \psi \circ \kappa$. This shows that the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is regular. Moreover, the group $K(Q)$ is an abelian p -group and since k is algebraically closed of characteristic p this implies that $H^q(K(Q); k^\times) = \{0\}$ for all $q > 0$. Thus 5.6 applies, proving the Theorem. \square

Proposition 6.2. *Let \mathcal{F} be a fusion system on a finite group P , let \mathcal{C} be a right ideal in \mathcal{F}^c and let $\bar{\mathcal{C}}$ be the canonical image of \mathcal{C} in the orbit category $\bar{\mathcal{F}}$. Let k be an algebraically closed field of characteristic p . The canonical functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ induces a regular functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ and an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\bar{\mathcal{C}}; k^\times)$.*

Proof. Two subgroups Q, P of P are isomorphic in \mathcal{F} if and only if they are isomorphic in $\bar{\mathcal{F}}$. Thus the canonical functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ sends non-isomorphisms to non-isomorphisms, hence induces a functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$, which in turn induces an isomorphism of partially ordered sets $[S(\mathcal{C})] \cong [S(\bar{\mathcal{C}})]$. Let $\sigma : [m] \rightarrow \mathcal{C}$ and $\tau : [n] \rightarrow \mathcal{C}$ be objects in $S(\mathcal{C})$ and denote by $\bar{\sigma}, \bar{\tau}$ their images in $S(\bar{\mathcal{C}})$. Let $(\alpha, \bar{\mu}) : \bar{\sigma} \rightarrow \bar{\tau}$ be a morphism in $S(\bar{\mathcal{C}})$; that is, $\alpha : [m] \rightarrow [n]$ is an order preserving map and $\bar{\mu} : \bar{\sigma} \cong \bar{\tau} \circ \alpha$ is a natural isomorphism. Explicitly, $\bar{\mu}$ consists of a compatible family of group isomorphisms $\bar{\mu}_i : \bar{\sigma}(i) \cong \bar{\tau}(\alpha(i))$; that is, for $0 \leq i < m$ we have $\bar{\mu}_{i+1} \circ \bar{\sigma}(i < i+1) = \bar{\tau}(\alpha(i) < \alpha(i+1)) \circ \bar{\mu}_i$. Since the functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$

is surjective on morphisms, there are group isomorphisms $\mu_i : \sigma(i) \rightarrow \tau(\alpha(i))$ in \mathcal{C} , but this family need not be a natural transformation. More precisely, for $0 \leq i < m$ there is a group element $v_i \in \tau(\alpha(i))$ such that if we denote by c_i the inner automorphism of $\tau(\alpha(i))$ given by conjugation with v_i , we have $c_{i+1} \circ \mu_{i+1} \circ \sigma(i < i+1) = \tau(\alpha(i) < \alpha(i+1)) \circ \mu_i$. An easy inductive argument shows that after replacing μ_{i+1} by $c_{i+1} \circ \mu_{i+1}$ we get a natural isomorphism $\mu : \sigma \cong \tau \circ \alpha$ lifting $\bar{\mu}$, which shows that the functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ is surjective on morphisms. Finally, if μ, μ' are two morphisms in $S(\mathcal{C})$ from σ to τ then by the regularity of $S(\mathcal{C})$ there is a unique automorphism β of σ such that $\mu' = \mu \circ \beta$. Thus, if in addition the images $\bar{\mu}, \bar{\mu}'$ of μ, μ' in $S(\bar{\mathcal{C}})$ are equal then the image $\bar{\beta}$ in $S(\bar{\mathcal{C}})$ satisfies $\mu = \mu \circ \beta$. Since every morphism in $S(\bar{\mathcal{C}})$ is a monomorphism this implies that $\bar{\beta} = \text{Id}_{\bar{\sigma}}$, and hence β belongs to the kernel $K(\sigma)$ of the canonical map from $\text{Aut}_{S(\mathcal{C})}(\sigma)$ to $\text{Aut}_{S(\bar{\mathcal{C}})}(\bar{\sigma})$, which shows that indeed the functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ is regular. Moreover, by standard properties of central group extensions, the group $K(\sigma)$ is a p -group. It follows that $H^q(K(\sigma); k^\times) = 0$ for positive q . Thus 5.6 implies $H^*(S(\mathcal{C}); k^\times) \cong H^*(S(\bar{\mathcal{C}}); k^\times)$, whence the stated isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\bar{\mathcal{C}}; k^\times)$. \square

Proposition 6.3. *Let \mathcal{F} be a fusion system on a finite p -group P having a centric linking system \mathcal{L} with structural functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$. Let \mathcal{C} be a right ideal in \mathcal{L} and let \mathcal{D} be its image in \mathcal{F} under π . Let k be an algebraically closed field of characteristic p . The functor π induces an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\mathcal{D}; k^\times)$.*

Proof. The centric linking system \mathcal{L} is an extension of \mathcal{F}^c by the centre functor $Z : \mathcal{F}^c \rightarrow \mathbb{Z}_{(p)}$ sending an \mathcal{F} -centric subgroup Q of P to its centre $Z(Q)$. Thus π induces a regular functor $\mathcal{C} \rightarrow \mathcal{D}$ and for any \mathcal{F} -centric subgroup of P we have $K(Q) = \ker(\text{Aut}_{\mathcal{L}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Q)) \cong Z(Q)$, hence $H^q(K(Q); k^\times) = \{0\}$ for $q > 0$. Thus 5.6 applies and yields the isomorphism as stated. \square

Proposition 6.4. *Let \mathcal{F} be a fusion system on a finite p -group P , let Q be a normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geq Q}$ be the right ideal in \mathcal{F} consisting of all subgroups of P containing Q . Let A be an abelian group. Restriction induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\text{Aut}_{\mathcal{F}}(Q); A)$.*

Proof. Let \mathcal{D} be the full subcategory of \mathcal{C} having Q as unique object and let $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ be the inclusion functor. For any subgroup R belonging to \mathcal{C} denote by $\iota_R : Q \subseteq R$ the inclusion homomorphism. For any morphism $\varphi : Q \rightarrow R$ we have $\varphi(Q) = Q$ because $\mathcal{F} = N_{\mathcal{F}}(Q)$. Thus composition with the inclusion morphism ι_R induces a bijection $\text{Aut}_{\mathcal{F}}(Q) \cong \text{Hom}_{\mathcal{F}}(Q, R)$. It follows from 3.2 that Φ has a right adjoint $\Psi : \mathcal{C} \rightarrow \mathcal{D}$. Then Ψ^* is right adjoint to Φ^* ; in other words, $\Phi_! = \Psi^*$ is exact, and hence we have $H^*(\mathcal{D}; A) = H^*(\mathcal{C}; \Psi^*(A)) = H^*(\mathcal{C}; A)$ by [9, 3.1]. \square

Proposition 6.5. *Let \mathcal{F} be a fusion system on a finite p -group P , let Q be an \mathcal{F} -centric normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geq Q}$ be the right ideal in \mathcal{F} consisting of all subgroups of P containing Q . Let $\mathcal{A} : \bar{\mathcal{F}}^c \rightarrow \mathbf{Ab}$ be a contravariant functor. Restriction induces an isomorphism $H^*((\bar{\mathcal{F}}^c)^{op}; \mathcal{A}) \cong H^*(\bar{\mathcal{C}}^{op}; \mathcal{A})$.*

Proof. The inclusion functor $\Psi : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{F}}^c$ has a left adjoint Φ sending R in $\bar{\mathcal{F}}^c$ to QR . Indeed, for any morphism $\varphi : R \rightarrow S$ in $\bar{\mathcal{F}}^c$ there is a morphism $\psi : QR \rightarrow QS$ in \mathcal{C} extending φ , and the image of ψ in $\bar{\mathcal{C}}$ is unique since every morphism in $\bar{\mathcal{F}}^c$ is an epimorphism, which shows that there is a canonical functor Φ sending R to QR . Moreover, for R in $\bar{\mathcal{F}}^c$ and S in \mathcal{C} we have $\text{Hom}_{\bar{\mathcal{C}}}(QR, S) \cong \text{Hom}_{\bar{\mathcal{F}}^c}(R, S)$, which shows that Φ is left adjoint to Ψ . Thus restriction along Ψ induces the stated isomorphism by [9, 3.1]. \square

Corollary 6.6. *Let \mathcal{F} be a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some \mathcal{F} -centric normal subgroup Q of P . Let L be a finite group with P as Sylow- p -subgroup such that $Q = O_p(L)$, $C_L(Q) = Z(Q)$ and such that $\mathcal{F}_P(L) = \mathcal{F}$. Then $H^*(\mathcal{F}^c; k^\times) \cong H^*(L; k^\times)$.*

Proof. We have $L/Z(Q) \cong \text{Aut}_{\mathcal{F}}(Q)$, and hence combining the above results yields $H^*(\mathcal{F}^c; k^\times) \cong H^*(\mathcal{C}; k^\times) \cong H^*(\text{Aut}_{\mathcal{F}}(Q); k^\times) \cong H^*(L; k^\times)$. \square

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